SOME s-NUMBERS OF AN INTEGRAL OPERATOR OF HARDY TYPE IN BANACH FUNCTION SPACES

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Abstract. Let $s_n(T)$ denote the nth approximation, isomorphism, Gelfand, Kolmogorov or Bernstein number of the Hardy-type integral operator T given by

$$Tf(x) = v(x)\int_a^x u(t)f(t)dt, \quad x \in (a,b) \ (-\infty < a < b < +\infty)$$

and mapping a Banach function space E to itself. We investigate some geometrical properties of E for which

$$C_1\int_a^b u(x)v(x)dx \le \limsup_{n\to\infty} ns_n(T) \le \limsup_{n\to\infty} ns_n(T) \le C_2\int_a^b u(x)v(x)dx$$

under appropriate conditions on u and v. The constants $C_1, C_2 > 0$ depend only on the space E.

1. Introduction

The s-numbers such as approximation, isomorphism, Bernstein, Gelfand and Kolmogorov numbers $s_n(T)$ of a compact linear map T acting between Banach spaces have proved to give a very useful measure of how compact the map is. For a fine survey of these numbers and their interactions with various parts of mathematics we refer to the monumental book [22] by Pietsch. The wealth of applications of these ideas has naturally led to the detailed study of s-numbers of particular maps, prominent among which are the weighted Hardy-type operators T, for which sharp upper and lower estimates of the approximation numbers in $L^p(a,b)$ spaces, $(1 \le p \le \infty)$ are investigated in [6] [7], [14], [15] and [21]. For various other s-numbers see [11] and [12] and the recent book [19]. When $v = u = 1$ (i.e. the non-weighted case) the problem of the estimation of approximation numbers for the Hardy operator acting between variable exponent Lebesgue spaces $L^{p(\cdot)}(a,b)$ was considered in [10]: see the recent books [19] and [13]. In Banach function spaces, estimates of approximation numbers were considered in [20].

Our purpose in this paper is to study s-numbers for a weighted Hardy-type operator T acting in a Banach function space E. Under some geometrical assumptions on E, and on the weights u, v, we obtain two-sided estimates for its approximation, isomorphism, Bernstein, Gelfand and Kolmogorov numbers. Our methods of proof

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are similar to those of [19] and are based on the extension of the estimates of the function $\mathcal{A}$ (see Section 4) to Banach function spaces under certain geometrical assumptions.

The paper is organized as follows. Section 2 contains notation, preliminaries and formulation of the main results, while in Section 3 we present an application to Lebesgue spaces with variable exponent and in Section 4 properties of the function $\mathcal{A}$ are established. Estimates of s-numbers of the operator are given in Section 5. Finally, asymptotic estimates and the proof of the main result are given in Section 6.

2. Notation, definitions and Preliminaries

Let $L(I)$ be the space of all Lebesgue-measurable real functions on $I = (a, b)$, where $-\infty < a < b < +\infty$. A Banach subspace E of $L(I)$ is said to be a Banach function space (BFS) if:

1) the norm $\|f\|_E$ is defined for every measurable function f and $f \in E$ if and only if $\|f\|_E < \infty :$ $\|f\|_E = 0$ if and only if $f = 0$ a.e.;

2) $\||f|\|_E = \|f\|_E$ for all $f \in E$;

3) if $0 \leq f \leq g$ a.e., then $\|f\|_E \leq \|g\|_E$;

4) if $0 \leq f_n \uparrow f$ a.e.,then $\|f_n\|_E \uparrow \|f\|_E$;

5) $L^\infty(I) \subset E \subset L^1(I)$.

Let J be an arbitrary interval of I. By $E(J)$ we denote the "restriction" of the space E to J; $E(J) = \{f\chi_J : \ f \in E\}$, with the norm $\|f\|_{E(J)} = \|f\chi_J\|_E$.

Given a Banach function space E, its associate space E' consists of those $g \in S$ such that $f \cdot g \in L^1$ for every $f \in E$ with norm $\|g\|_{E'} = \sup\{\|f \cdot g\|_{L^1} : \ \|f\|_E \leq 1\ \}$. E' is a BFS on I and a closed norm fundamental subspace of the conjugate space E^*.

We say that the space E has absolutely continuous norm (AC-norm) if for all $f \in E$, $\|f\chi_{X_n}\|_E \to 0$ for every sequence of measurable sets $\{X_n\} \subset I$ such that $\chi_{X_n} \to 0$ a.e. Note that the Hölder inequality

$$\int_I f(x)g(x)dx \leq \|f\|_E \|g\|_{E'}$$

holds for all $f \in E$ and $g \in E'$ and is sharp (for more details we refer to [1]).

Let E be a Banach space with dual E^*; the value of x^* at $x \in E$ is denoted by $(x, x^*)_X$ or (x, x^*) .

We recall that E is said to be strictly convex if whenever $x, y \in E$ are such that $x \neq y$ and $\|x\| = \|y\| = 1$, and $\lambda \in (0, 1)$, then $\|\lambda x + (1\lambda)y\| < 1$. This simply means that the unit sphere in E does not contain any line segment.

By Π we denote the family of all sequences $\mathcal{Q} = \{I_i\}$ of disjoint intervals in I such that $I = \cup_{I_i \in \mathcal{Q}} I_i$. We ignore the difference in notation caused by a null set.

Everywhere in the sequel by $l_{\mathcal{Q}}$, $(\mathcal{Q} \in \Pi)$ we denote a Banach sequence space (BSS) (indexed by a partition $\mathcal{Q} = \{I_i\}$ of I), meaning that axioms 1)-4) are satisfied with respect to the counting measure, and let $\{e_{I_i}\}$ denote the standard unit vectors in $l_{\mathcal{Q}}$.

Throughout the paper we denote by C, C_1, C_2 various positive constants independent of appropriate quantities and not necessarily the same at each occurrence. By $A \approx B$ we mean that $0 < C_1 \leq A/B \leq C_2 < \infty$ for some C_1, C_2.

Definition 2.1. Let $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q} \in \Pi}$ be a family of BSSs. A BFS E is said to satisfy a uniform upper (lower) $l-$estimate if there exists a constant $C > 0$ such that for

every $f \in E$ and $\mathcal{Q} \in \Pi$ we have

$$\|f\|_E \le C \| \sum_{I_i \in \mathcal{Q}} \|f\chi_{I_i}\|_E \cdot e_{I_i}\|_{l_{\mathcal{Q}}} \quad \left(\| \sum_{I_i \in \mathcal{Q}} \|f\chi_{I_i}\|_E \cdot e_{I_i}\|_{l_{\mathcal{Q}}} \le C\|f\|_E \right).$$

Definition 2.1 was introduced in [16]. The idea behind it is simply to generalize the following property of the Lebesgue norm:

$$\|f\|_{L^p}^p = \sum_i \|f\chi_{\Omega_i}\|_{L^p}^p$$

for a partition of $\mathbb{R}^n$ into measurable sets Ω_i. The notions of uniform upper (lower) $l-$estimates, when $l_{\mathcal{Q}_1} = l_{\mathcal{Q}_2}$ for all $\mathcal{Q}_1, \mathcal{Q}_2 \in \Pi$, were introduced by Berezhnoi in [2].

Note that if a BFS E simultaneously satisfies upper and lower $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\Pi}$ estimates, then there exists a constant $C > 0$ such that, for any $f \in E$ and $\mathcal{Q} \in \Pi$,

$$\frac{1}{C}\|f\|_E \le \left\| \sum_{I_i \in \mathcal{Q}} \frac{\|f\chi_{I_i}\|_E}{\|\chi_{I_i}\|_E} \cdot \chi_{I_i} \right\|_E \le C\|f\|_E. \tag{2.1}$$

Note also that if E simultaneously satisfies upper and lower $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\Pi}$ estimates then E' simultaneously satisfies upper and lower $l' = \{l'_{\mathcal{Q}}\}_{\mathcal{Q}\in\Pi}$ estimates (see [16]).

We investigate properties of the Hardy-type operator of the form

$$Tf(x) = T_{a,I,u,v}f(x) = v(x)\int_a^x f(t)u(t)dt,$$

where u and v are given real valued nonnegative functions with $|\{x : u(x) = 0\}| = |\{x : v(x) = 0\}| = 0$ as a mapping between BFS (by $|\cdot|$ we denote Lebesgue measure). This operator appears naturally in the theory of differential equations and it is important to establish when operators of this kind have properties such as boundedness, compactness, and to estimate their eigenvalues, or their approximation numbers. We shall assume that

$$u\chi_{(a,x)} \in E' \tag{2.2}$$

and

$$\chi_{(x,b)} \in E \tag{2.3}$$

whenever $a < x < b$.

In [16] the following was proved.

Theorem 2.2. *Let E and F be BFSs with the following property: there exists a family of BSS $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\Pi}$ such that E satisfies a uniform lower l-estimate and F a uniform upper l-estimate. Suppose that* (2.2) *and* (2.3) *hold. Then T is a bounded operator from E into F if and only if*

$$\sup_{a<t<b} A(t) = \sup_{a<t<b} \|v\chi_{(t,b)}\|_F \|u\chi_{(a,t)}\|_{E'} < \infty.$$

We observe that similar results hold when we replace v and u by $v\chi_J$ and $u\chi_J$ respectively, where J is any subinterval of I. Note that in [16] the verification of the above conditions is carried only for I. However, the methods of proof work equally well for arbitrary intervals $J \subset I$.

Theorem 2.3. *Let $J = (c, d)$ be any interval of I; let E and F be BFS for which there exists a family of BSS $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\Pi}$ such that E satisfies a uniform lower l-estimate and F a uniform upper l-estimate. Then the operator*

$$T_J f(x) = v(x)\chi_J(x) \int_a^x u(t)\chi_J(x)f(t)dt$$

is bounded from E into F if and only if

$$A_J = \sup_{t\in J} A_J(t) = \sup_{t\in J} \|v\chi_J\chi_{(t,d)}\|_F \|u\chi_J\chi_{(c,t)}\|_{E'} < \infty.$$

Moreover $A_J \leq \|T_J\| \leq K \cdot A_J$, where $K \geq 1$ is a constant independent of J.

In [8] the authors establish a general criterion for T to be compact from E to F when $T : E \to F$ is bounded. Indeed the following theorem is valid.

Theorem 2.4. *Let $T : E \to F$ be bounded, where E, F are BFS with AC- norms. Then T is compact from E to E if and only if the following two statements are satisfied:*

$$\lim_{x\to a+} \sup_{a<r<x} \|v\chi_{(r,x)}\|_F \|u\chi_{(a,r)}\|_{E'} = 0,$$

and

$$\lim_{x\to b-} \sup_{x<r<b} \|v\chi_{(r,b)}\|_F \|u\chi_{(x,r)}\|_{F'} = 0.$$

Note that if E and F have AC-norms and $u \in E'$, $v \in F$ then $T : E \to F$ is compact.

More detailed information about the compactness properties of T is provided by the approximation, isomorphism, Bernstein, Gelfand and Kolmogorov numbers and we next recall the definition of those quantities.

$B(E, F)$ will denote the space of all bounded linear maps of E to F. Given a closed linear subspace M of E, the embedding map of M into E will be denoted by J_M^E and the canonical map of E onto the quotient space E/M by Q_M^E. Let $S \in B(E, E)$. Then the modulus of injectivity of T is

$$j(S) = \sup\{\rho \geq 0 : \ \|Sx\|_E \geq \rho\|x\|_E \ \text{ for all } \quad x \in E\}.$$

Definition 2.5. Let $S \in B(E, E)$ and $n \in \mathbb{N}$. Then the nth approximation, isomorphism, Gelfand, Bernstein and Kolmogorov numbers of S are defined by

$$a_n(S) = \inf\{\|S - P\| : \ P \in B(E, E), \ \text{rank}(P) < n\};$$

$$i_n(S) = \sup\{\|A\|^{-1}\|B\|^{-1}\},$$

where the supremum is taken over all possible Banach spaces G with $\text{dim}G \geq n$ and maps $A \in B(E, G), \ B \in B(G, E)$ such that ASB is the identity on G;

$$c_n(S) = \inf\{\|SJ_M^E\| : \ \text{codim}(M) < n\};$$

$$b_n(S) = \sup\{j(SJ_M^E); \ \dim(M) \geq n\};$$

$$d_n(S) = \inf\{\|Q_M^E S\| : \ \dim(M) < n\}.$$

respectively.

Below $s_n(S)$ denotes any of the nth approximation, isomorphism, Gelfand, Kolmogorov or Bernstein numbers of the operator S. We summaries some of the facts concerning the numbers $s_n(S)$ in the following theorem (see [19]):

Theorem 2.6. *Let $S \in B(E,E)$ and $n \in \mathbb{N}$. Then*

$$a_n(S) \geq c_n(S) \geq b_n(S) \geq i_n(S)$$

and

$$a_n(S) \geq d_n(S) \geq i_n(S).$$

The behavior of the s-numbers of the Hardy-type operator T is reasonably well understood in case $E = F = L^p(a,b)$.

Theorem 2.7. *Suppose that $1 < p < \infty$, $v \in L^p(a,b)$, $u \in L^q(a,b)$ where $1/p + 1/q = 1$. Then for $T : L^p(a,b) \to L^p(a,b)$ we have*

$$\lim_{n\to\infty} n s_n(T) = \frac{1}{2}\gamma_p \int_a^b u(x)v(x)dx,$$

where $\gamma_p = \pi^{-1}p^{1/q}q^{1/p}\sin(\pi/p)$.

When $p = 2$ and the s_n are approximation numbers this was first established in [7], see also [21]. The general case, namely that when $1 < p < \infty$, was proved in [15], where it appears as a special case of results for trees. When $p = 2$, for nice u and v these results were improved in [9] and more recently extended for $1 < p < \infty$ in [18].

We say that a space E fulfills the Muckenhoupt condition if for some constant $C > 0$ and for all intervals $J \subset I$ we have

$$\frac{1}{|J|}\|\chi_J\|_E\|\chi_J\|_{E'} \leq C.$$

Note that if E fulfills the Muckenhoupt condition, then using Hölders inequality we obtain

$$\frac{1}{|J|}\int_J |f(x)|dx \leq C\frac{\|f\chi_J\|_E}{\|\chi_J\|_E},$$

and if additionally E simultaneously satisfies upper and lower $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\Pi}$ estimates, then from (2.1) we obtain

$$\left\| \sum_{I_i\in\mathcal{Q}} \frac{1}{|I_i|}\int_{I_i} |f(x)|dx \right\|_E \leq C_1\|f\|_E,$$

where $C_1 > 0$ is an absolute constant independent of the partition $\mathcal{Q}$ of I. If for a space E we have the Muckenhoupt condition and (2.1), we denote this by writing $E \in \mathcal{M}$. Note that in the case of a reflexive variable exponent Lebesgue space the condition $L^{p(\cdot)} \in \mathcal{M}$ implies the boundedness of the Hardy-Littlwood maximal operator in $L^{p(\cdot)}$ (see [3], [4]).

The main result of this paper is the following theorem.

Theorem 2.8. *Let E be BFS belong to the class $\mathcal{M}$. Let the spaces E, E^* be strictly convex and assume that E and E' have AC- norms. Suppose $u \in E'$, $v \in E$. Then there exists constants $C_1 = C_1(E), C_2 = C_2(E) > 0$ such that, for the map $T : E \to E$*

$$C_1\int_a^b u(x)v(x)dx \leq \limsup_{n\to\infty} n s_n(T) \leq \limsup_{n\to\infty} n s_n(T) \leq C_2\int_a^b u(x)v(x)dx.$$

3. Variable exponent Lebesgue spaces

Given a measurable function $p(\cdot) : (a,b) \to [1,+\infty)$, $L^{p(\cdot)}(a,b)$ denotes the set of measurable functions f on (a,b) such that for some $\lambda > 0$,

$$\int_{(a,b)} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{(a,b)} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

These spaces and corresponding variable Sobolev spaces $W^{k,p(\cdot)}$ are of interest in their own right, and also have applications to partial differential equations and the calculus of variations.(For more details of results about variable exponent Lebesgue spaces we refer to [3] and [4]).

We say that a function $p : (a,b) \to (1,\infty)$ is log-Hölder continuous if there exists $C > 0$ such that

$$|p(x) - p(y)| \le \frac{C}{\log(e + 1/|x-y|)} \quad \text{for all} \quad x, y \in (a,b) \quad \text{and} \quad x \ne y.$$

Denote by $\mathcal{P}_{\log}$ the set of all log-Hölder continuous exponents that satisfy

$$p_- = \text{ess inf}_{x\in(a,b)} p(x) > 1, \quad p_+ = \text{ess sup}_{x\in(a,b)} p(x) < \infty.$$

Note that the log-Hölder continuous condition is in fact optimal in the sense of the modulus of continuity, for boundedness of the Hardy-Littlewood maximal operator in variable Lebesgue spaces (see [3], [4]).

We say that a exponent $p(\cdot) \in \mathcal{P}_{\log}$ is strongly log-Hölder continuous (and write $p(\cdot) \in \mathcal{SP}_{\log}$) if there is an increasing continuous function defined on $[0, b-a]$ such that $\lim\limits_{t\to 0+} \psi(t) = 0$ and

$$-|p(x) - p(y)| \ln |x-y| \le \psi(|x-y|) \ \text{ for all } x, y \in (a,b) \ \text{ with } 0 < |x-y| < 1/2.$$

In [16] the following was proved.

Proposition 3.1. *Let $p(\cdot) \in \mathcal{P}_{\log}$. Then $L^{p(\cdot)}(a,b) \in \mathcal{M}$.*

Note that there exists another classes of exponents giving rise to property (2.1). Indeed, let $p(\cdot) : [0,1] \to [1,+\infty)$ be log-Hölder continuous, $w(t) = \int_a^t l(u)du$, $t \in (a,b)$, $w(b) = 1$, $l(u) > 0$ $(u \in (a,b))$. Then $L^{p((w(\cdot))}(a,b)$ has property (2.1)) (see [17]).

From Theorem 2.8 and Proposition 3.1 we obtain

Corollary 3.2. *Let $p(\cdot) \in \mathcal{P}_{\log}$ and $v \in L^{p(\cdot)}(a,b)$, $u \in L^{q(\cdot)}(a,b)$ $(1/p(x) + 1/q(x) = 1,\ x \in (a,b))$. Then T acts from the variable exponent space $L^{p(\cdot)}(a,b)$ to itself and*

$$C_1 \int_{(a,b)} u(x)v(x)dx \le \liminf_{n\to\infty} n s_n(T) \le \limsup_{n\to\infty} n s_n(T) \le C_2 \int_{(a,b)} u(x)v(x)dx.$$

An analogue of Theorem 2.8 in the setting of spaces with variable exponent when $u = v = 1$ was investigated in [10], where the following theorem was proved.

Theorem 3.3. *Let $p(\cdot) \in \mathcal{SP}_{\log}$ and $u = v = 1$. Then T acts from the variable exponent space $L^{p(\cdot)}(a, b)$ to itself and*

$$\lim_{n\to\infty} ns'_n(T) = \frac{1}{2\pi}\int_I (q(x)p(x)^{p(x)-1})^{1/p(x0} \sin(\pi/p(x))dx,$$

where $s'_n(T)$ stands for any of the n-th approximation, Gelfand, Kolmogorov and Bernstein numbers of T.

4. Properties of $\mathcal{A}$

Here we establish properties of the function $\mathcal{A}$ which we shall need in the next section.

Definition 4.1. Let E be a BFS, J be a subinterval of $I = (a, b)$, $c \in [a, b]$, and suppose that $u \in E'(J)$ and $v \in E(J)$. We define

$$\mathcal{A}(J) = \mathcal{A}(J, u, v) = \sup_{f\in E,\, f\neq 0} \inf_{\alpha\in\mathbb{R}} \frac{\|T_{c,J}f - \alpha v\|_{E(J)}}{\|f\|_{E(J)}},$$

where

$$T_{c,J}f(x) = v(x)\chi_J(x)\int_c^x f(t)u(t)\chi_J(t)dt.$$

We prove some basic properties of $\mathcal{A}(J)$. Choosing $\alpha = 0$ we immediately obtain

$$\mathcal{A}(J) \le \|T_{c,J}\| \le K \cdot A_J.$$

Note that for $d \in [a, b]$,

$$T_{d,J}f(x) = T_{c,J}f(x) + v(x)\chi_J(x)\int_d^c f(t)u(t)\chi_J(t)dt$$

and the number $\mathcal{A}(J, u, v)$ is independent of $c \in [a, b]$.

Lemma 4.2. *Let E be a BFS, J be a subinterval of I, and suppose that $u \in E'(J)$ and $v \in E(J)$. Set*

$$\widetilde{\mathcal{A}}(J) = \sup_{\|f\|_{E(J)}=1} \inf_{|\alpha|\le 2\|u\|_{E'(J)}} \|T_{c,J}f - \alpha v\|_{E(J)}.$$

Then $\mathcal{A}(J) = \widetilde{\mathcal{A}}(J)$.

Proof. Hölder's inequality yields

$$\|T_{c,J}\| \le \|u\chi_J\|_{E'(J)}\|v\chi_J\|_{E(J)}.$$

Let $\|f\|_{E(J)} = 1$ and $|\alpha| > 2\|u\|_{E'(J)}$. Then $|\alpha| > \frac{2\|T_{c,J}\|}{\|v\|_{E(J)}}$ and using the triangle inequality we obtain

$$\begin{aligned}\|\alpha v - T_{c,J}f\|_{E(J)} &\ge |\alpha|\|v\|_{E(J)} - \|T_{c,J}\|\|f\|_{E(J)}\\ &> 2\|T_{c,J}\| - \|T_{c,J}\|\\ &= \|T_{c,J}\|.\end{aligned}$$

We have

$$\begin{aligned}&\|T_{c,J}\|\\ &\quad\ge \mathcal{A}(J)\\ &\quad= \sup_{\|f\|_{E(J)}=1} \min\{\inf_{|\alpha|\le 2\|u\|_{E'(J)}} \|T_{c,J}f - \alpha v\|_{E(J)}, \inf_{|\alpha|>2\|u\|_{E'(J)}} \|T_{c,J}f - \alpha v\|_{E(J)}\}\end{aligned}$$

$$= \sup_{\|f\|_{E(J)}=1} \inf_{|\alpha|\le 2\|u\|_{E'(J)}} \|T_{c,I}f - \alpha v\|_{E(J)} = \widetilde{\mathcal{A}}(J).$$

□

Note that using the same arguments we may prove that

$$\mathcal{A}(J) = \sup_{\|f\|_{E(J)}\le 1} \inf_{|\alpha|\le 2\|u\|_{E'(J)}} \|T_{c,J}f - \alpha v\|_{E(J)}.$$

Lemma 4.3. *Let E be a BFS and the dual E^* of the space E has AC- norm. Let J be a subinterval of I, and suppose that $u \in E'(J)$ and $v \in E(J)$. Then:*

1. *The function $\mathcal{A}(x,d)$ is non-increasing and continuous on (c,d).*
2. *The function $\mathcal{A}(c,x)$ is non-decreasing and continuous on (c,d).*
3. $\lim_{x\to c-} \mathcal{A}(c,x) = \lim_{x\to d+} \mathcal{A}(x,d) = 0.$

Proof. That $\mathcal{A}(x,d)$ is non-increasing is easy to see. Fix $y, c < y < d$. Let $\varepsilon > 0$. Fix $h_0 > 0$ such that $y - h_0 > 0$ and $\|u\|_{E'(y-h,y)} < \varepsilon$ for $0 < h \le h_0$.

Let $D_h = \|u\|_{E'(y-h,d)}$ $(0 \le h \le h_0)$ and $w(y) = \chi_{(y,d)} \int_{y-h}^{y} f(t)u(t)dt$.

We have

$$\begin{aligned}
\mathcal{A}(y,d) &\le \mathcal{A}(y-h,d)\\
&= \sup_{\|f\|_{E(y-h,d)}=1} \inf_{\alpha\in\mathbb{R}} \|\alpha v - T_{y-h,(y-h,d)}f\|_{E((y-h,d))}\\
&= \sup_{\|f\|_{E(y-h,d)}=1} \inf_{|\alpha|\le 2D_h} \{\|(\alpha v - T_{y-h,(y-h,d)}f)\chi_{(y-h,y)}\|_{E((y-h,y))}\\
&\qquad + \|(\alpha v - T_{y-h,(y-h,d)}f)\chi_{(y,d)}\|_{E((y,d))}\}\\
&\le \sup_{\|f\|_{E(y-h,d)}=1} \inf_{|\alpha|\le 2D_h} \{\|T_{y-h,(y-h,y)}|E((y-h,y)) \to E((y-h,y))\|\times\\
&\qquad \times \|f\|_{E((y-h,y))} + \|(\alpha v - T_{y,(y-h,d)}f - vw)\chi_{(y,d)}\|_{E((y,d))}\}\\
&\le \sup_{\|f\|_{E(y-h,d)}=1} \inf_{|\alpha|\le 2D_h} \Big\{\|u\|_{E'((y-h,y))}\|v\|_{E((y-h,y))}\|f\|_{E((y-h,y))}\\
&\qquad + \|v\|_{E((y,d))}\|u\|_{E'((y-h,y))}\|f\|_{E((y-h,y))}\\
&\qquad + \|(\alpha v - T_{y,(y,d)}f)\chi_{(y,d)}\|_{E((y,d))}\Big\}\\
&\le \|v\|_{E((y-h,y))}\varepsilon + \|v\|_{E((y,d))}\varepsilon\\
&\qquad + \sup_{\|f\|_{E(y-h,d)}=1} \inf_{|\alpha|\le 2D_h} \|T_{y,(y,d)}f - \alpha v\|_{E((y,d))}.
\end{aligned}$$

Since $D_0 \le D_h \le D_{h_0}$ we have

$$\begin{aligned}
\sup_{\|f\|_{E((y-h,d))}=1} \inf_{|\alpha|\le 2D_h} \|T_{y,(y,d)}f - \alpha v\|_{E((y,d))}
&\le \sup_{\|f\|_{E((y-h,d))}=1} \inf_{|\alpha|\le 2D_0} \|T_{y,(y,d)}f - \alpha v\|_{E((y,d))}\\
&= \sup_{\|f\|_{E((y,d))}\le 1} \inf_{|\alpha|\le 2D_0} \|T_{y,(y,d)}f - \alpha v\|_{E((y,d))}\\
&= \mathcal{A}(y,d)
\end{aligned}$$

and thus

$$\mathcal{A}(y,d) \le \mathcal{A}(y-h,d) \le \|v\|_{E((y-h,y))}\varepsilon + \|v\|_{E((y,d))}\varepsilon + \mathcal{A}(y,d),$$

which proves that

$$\lim_{h\to 0+} \mathcal{A}(y-h,d) = \mathcal{A}(y,d).$$

Analogously

$$\lim_{h\to 0+} \mathcal{A}(y+h,d) = \mathcal{A}(y,d).$$

In the same way we prove 2 and 3, which finishes the proof of the lemma. □

Lemma 4.4. *Let E be a BFS satisfying the condition* (2.1) *and suppose that E' has AC-norm. Let $J = (c,d)$ be a subinterval of I, and suppose that $u \in E'(J)$ and $v \in E(J)$. Then*

$$\mathcal{A}(J) \le \inf_{x\in J} \|T_{x,J}\,|E(J) \to E(J)\|. \tag{4.1}$$

The norms $\|T_{x,J}\|$, $\|T_{x,(c,x)}\|$, $\|T_{x,(x,d)}\|$ of the operators $T_{x,J}$ $T_{x,(c,x)}$, $T_{x,(x,d)}$, from $E(J)$ to $E(J)$, are continuous in $x \in (c,d)$ and there exists $e \in J$ such that

$$\|T_{e,(c,e)}\| = \|T_{e,(e,d)}\|. \tag{4.2}$$

For any $x \in J$

$$\|T_{x,J}\| \approx \max\{\|T_{x,(c,x)}\|,\ \|T_{x,(x,d)}\|\}, \tag{4.3}$$

and

$$\min_{x\in J} \|T_{x,J}\| \approx \|T_{e,J}\|. \tag{4.4}$$

Proof. For any $x \in (c,d)$,

$$\mathcal{A}(J) \le \sup\{\|T_{x,J}f\|_{E(J)} : \|f\|_{E(J)} = 1\} = \|T_{x,J}|\,E(J) \to E(J)\|,$$

and consequently we have (4.1).

To prove the continuity of $\|T_{x,(x,d)}\|$, we first note that for $z, y \in (c,d), z < y$,

$$\begin{aligned} T_{z,(z,d)}f(x) - T_{y,(y,d)}f(x) &= v(x)\chi_{(y,d)}(x)\int_z^y f(t)u(t)dt \\ &\quad + v(x)\chi_{(z,y)}(x)\int_z^y f(t)u(t)dt. \end{aligned}$$

Hence, applying Hölder's inequality,

$$\|T_{z,(z,d)} - T_{y,(y,d)}\| \le \|v\|_{E((y,d))}\|u\|_{E'((z,y))} + \|v\|_{E((z,y))}\|u\|_{E'((z,y))}$$

and so

$$\big|\|T_{z,(z,d)}\| - \|T_{y,(y,d)}\|\big| \le \|T_{z,(z,d)} - T_{y,(y,d)}\| \le 2\|u\|_{E'((z,y))}\|v\|_{E((z,d))},$$

which yields the continuity of $\|T_{x,(x,d)}\|$. Similarly we obtain the continuity for $\|T_{x,(c,x)}\|$ and $\|T_{x,J}\|$.

If $\operatorname{supp} f \subset (y,d)$ then for $z < y$,

$$T_{z,(z,d)}f(x) = T_{y,(y,d)}f(x).$$

Consequently

$$\|T_{y,(y,d)}\| \le \|T_{z,(z,d)}\|$$

and similarly

$$\|T_{z,(c,z)}\| \le \|T_{y,(c,y)}\|.$$

The identity (4.2) follows from these inequalities and the continuity of the norms $\|T_{x,(c,x)}\|, \|T_{x,(x,d)}\|$.

Let $f \in E(J)$ and set $f_1 = f\chi_{(c,x)},\ f_2 = f\chi_{(x,d)}$. Then

$$(T_{x,J}f)(t) = (T_{x,(c,x)}f_1)(t) + (T_{x,(x,d)}f_2)(t).$$

We have

$$\begin{aligned}\|T_{x,J}f\|_{E(J)} &\approx \max\{\|T_{x,(c,x)}f_1\|_{E((c,x))}, \|T_{x,(x,d)}f_2\|_{E((x,d))}\}\\ &\leq C\max\{\|T_{x,(c,x)}\|, \|T_{x,(x,d)}\|\}\max\{\|f_1\|_{E((c,x))}, \|f_2\|_{E((x,d))}\}\\ &\leq C\max\{\|T_{x,(c,x)}\|, \|T_{x,(x,d)}\|\}\|f\|_{E(J)}.\end{aligned}$$

Consequently

$$\|T_{x,J}\| \leq C\max\{\|T_{x,(c,x)}\|, \|T_{x,(x,d)}\|\}.$$

The reverse inequality is obvious and (4.3) is proved. From (4.2), (4.3) and the above analysis, we have (4.4). □

Definition 4.5. Let E be a BFS satisfying the condition (2.1) and suppose that E' has AC-norm. Let $J = (c, d)$ be a subinterval of I, and suppose that $u \in E'(J)$ and $v \in E(J)$. Define

$$\widehat{\mathcal{A}}(J) = \|T_{e,(c,e)}\|$$

where $e \in J$ defined by 4.2.

Lemma 4.6. *Let E be a BFS satisfying the condition* (2.1) *and suppose that E' has AC-norm; let J be a subinterval of I, and suppose that $u \in E'(J)$ and $v \in E(J)$. Then*

1, $\|T_{x,(c,x)}\|$ *is strictly increasing and* $\|T_{x,(x,d)}\|$ *is strictly decreasing on* (c, d).
2. $\widehat{\mathcal{A}}(c, x)$ *is strictly increasing and* $\widehat{\mathcal{A}}(x, d)$ *is strictly decreasing on* (c, d).

Proof. The strictly monotonic properties of the functions $\|T_{x,(c,x)}\|$ and $\widehat{\mathcal{A}}(c, x)$ follow from the condition $|\{x : u(x) = 0\}| = |\{x : v(x) = 0|\|| = 0$. If we use arguments analogous to those in the proof of Lemma 4.4 we may prove continuity of $\widehat{\mathcal{A}}(c, x)$. □

Lemma 4.7. *Let E be a strictly convex BFS. Then given any f, $e \in E$, $e \neq 0$ there is a unique scalar c_f such that*

$$\|f - c_f e\|_E = \inf_{c\in\mathbb{R}} \|f - ce\|_E.$$

Proof. Since $\|f - ce\|_E$ is continuous in c and tends to ∞ as $c \to \infty$, the existence of c_f is guaranteed by the local compactness of $\mathbb{R}$. The uniqueness of c_f follows from the strict convexity of E. □

Lemma 4.8. *Let E be a strictly convex BFS and given $f \in E$, let c_f be the unique scalar such that $\|f - c_f e\|_E = \inf_{c\in\mathbb{R}} \|f - ce\|_E$, for $e \neq 0$, $e \in E$. Then the map $f \mapsto c_f$ is continuous.*

Proof. Suppose $\|f_n - f\|_E \to 0$. Since c_{f_n} is bounded, we may suppose that $c_{f_n} \to c$. Then

$$\|f_n - c_f e\|_E \geq \|f_n - c_{f_n} e\|_E$$

and so

$$\|f - c_f e\|_E \geq \|f - ce\|_E$$

which gives $c = c_f$. □

Lemma 4.9. *Let E be a strictly convex BFS satisfying the condition* (2.1) *and suppose that E' has AC-norm. Let $J = (c, d)$ be a subinterval of I, and suppose that $u \in E'(J)$ and $v \in E(J)$. Then*

$$\mathcal{A}(J) \approx \min_{x \in J} \|T_{x,J}|E(J) \to E(J)\| \approx \|T_{e,J}|E(J) \to E(J)\|, \tag{4.5}$$

where $e \in I$ defined by (4.2)*.*

Proof. Note that (using (4.3) and (4.4))

$$\begin{aligned} \|T_{e,(c,e)}|E(J) \to E(J)\| &= \|T_{e,(e,d)}|E(J) \to E(J)\| \\ &\leq \|T_{e,J}|E(J) \to E(J)\| \\ &\leq C_1 \|T_{e,(c,e)}|E(J) \to E(J)\|. \end{aligned} \tag{4.6}$$

Let $\alpha < \|T_{e,J}\|$. Set $T_{e,J} = vF$, where,

$$Ff(x) = F_{e,J}f(x) = \chi_J(x) \int_e^x f(t)u(t)\chi_J(t)dt.$$

By (4.6) it follows that there exists f_i, $i = 1, 2$, supported in (c, e) and (e, d), respectively, such that $\|f_i\|_E = 1$, $\|T_{e,J}f_i\|_{E(J)} > \alpha/C_1$ and f_1 positive, f_2 negative. Note that the same is true of the signs of the corresponding values of c_{vFf_1}, c_{vFf_2}, with $e = v$ (see Lemma 4.7-4.8). Hence by the continuity established in Lemma 4.8, there is a $\lambda \in (0, 1)$ such that $c_{vFg} = 0$ for $g = \lambda f_1 + (1 - \lambda) f_2$.

We have

$$\begin{aligned} \|T_{e,J}g\|_{E(J)} &\geq C_2 \max\{\|\lambda T_{e,(c,e)}f_1\|_{E((c,e))}, \|(1-\lambda)T_{e,(e,d)}f_2\|_{E((e,d))}\} \\ &\geq C_3 \alpha \|g\|_{E(J)}. \end{aligned}$$

We have

$$\mathcal{A}(J) \geq \inf_{\alpha \in \mathbb{R}} \|vFg - \alpha v\|_{E(J)}/\|g\|_{E(J)} = \|vFg\|_{E(J)}/\|g\|_{E(J)} \geq C_3 \alpha.$$

Since $\alpha < \|T_{e,J}\|$ is arbitrary, $\mathcal{A}(J) \geq C_3 \|T_{e,J}\|$. and the first equivalence follows from (4.1). Using (4.4), we obtain the second equality of (4.5). $\square$

Lemma 4.10. *Let $J = (c, d)$ be a subinterval of I, and suppose that u_1, u_2 belong to $E'(J)$ and $v \in E(J)$. Then*

$$|\mathcal{A}(J, u_1, v) - \mathcal{A}(J, u_2, v)| \leq \|u_1 - u_2\|_{E'(J)} \|u\|_{E(J)}.$$

Proof.

$$\begin{aligned} \mathcal{A}(J, u_1, v) &= \sup_{\|f\|_{E(J)}=1} \inf_{\alpha \in \mathbb{R}} \left\| v(x) \left(\int_a^x f(t)(u_1(t) - u_2(t) + u_2(t))dt - \alpha \right) \right\|_{E(J)} \\ &\leq \sup_{\|f\|_{E(J)}=1} \inf_{\alpha \in \mathbb{R}} \left[\left\| v(x) \int_a^x f(t)(u_1(t) - u_2(t))dt \right\|_{E(J)} \right. \\ &\qquad \left. + \left\| v(x) \int_a^x f(t)u_2(t)dt - \alpha v(x) \right\|_{E(J)} \right] \\ &\leq \sup_{\|f\|_{E(J)}=1} \inf_{\alpha \in \mathbb{R}} \left[\|u_1 - u_2\|_{E'(J)} \|u\|_{E(J)} \right. \\ &\qquad \left. + \left\| v(x) \int_a^x f(t)u_2(t)dt - \alpha v(x) \right\|_{E(J)} \right] \end{aligned}$$

$$\leq \|u_1 - u_2\|_{E'(J)} \|u\|_{E(J)} + \mathcal{A}(J, u_2, v).$$

The same holds with u_1 and u_2 interchanged, and the result follows. □

Lemma 4.11. *Let $J = (c, d)$ be a subinterval of I, and suppose that $u \in E'(I)$ and $v_1,\ v_2 \in E(I)$. Then*

$$|\mathcal{A}(J, u, v_1) - \mathcal{A}(J, u, v_2)| \leq 3\|v_1 - v_2\|_{E(J)} \|u\|_{E'(J)}.$$

Proof. Let

$$T_J^1 f(x) = v_1(x)\chi_J(x) \int_a^x f(t)u(t)dt,$$
$$T_J^2 f(x) = v_2(x)\chi_J(x) \int_a^x f(t)u(t)dt,$$
$$T_I^3 f(x) = (v_1(x) - v_2(x))\chi_J(x) \int_a^x f(t)u(t)dt$$

Suppose that $\mathcal{A}(J, u, v_1) > \mathcal{A}(J, u, v_2)$. By Lemma 4.2 we have

$$\begin{aligned}
&\mathcal{A}(J, u, v_1) - \mathcal{A}(J, u, v_2) \\
&\quad = \sup_{\|f\|_{E(J)}=1} \inf_{\alpha \in \mathbb{R}} \|T_J^1 f - \alpha v_1\|_{E(J)} - \mathcal{A}(J, u, v_2) \\
&\quad = \sup_{\|f\|_{E(J)}=1} \inf_{|\alpha|2 \leq \|u\|_{E(J)}} \|T_J^1 f - \alpha v_1\|_{E(J)} - \mathcal{A}(J, u, v_2) \\
&\quad \leq \sup_{\|f\|_{E(J)}=1} \inf_{|\alpha| \leq 2\|u\|_{E(J)}} \left(\|T_J^3 f - \alpha(v_1 - v_2)\|_{E(J)} + \|T_J^2 f - \alpha v_2\|_{E(J)}\right) \\
&\qquad\quad - \mathcal{A}(J, u, v_2) \\
&\quad \leq \sup_{\|f\|_{E(J)}} \inf_{|\alpha| \leq \|u\|_{E(J)}} \left(3\|v_1 - v_2\|_{E(J)} \|u\|_{E'(J)} + \|T_J^2 - \alpha v_2\|_{E(J)}\right) \\
&\qquad\quad - \mathcal{A}(J, u, v_2) \\
&\quad \leq 3\|v_1 - v_2\|_{E(J)} \|u\|_{E'(J)} + \mathcal{A}(J, u, v_2) - \mathcal{A}(J, u, v_2) \\
&\quad = 3\|v_1 - v_2\|_{E(J)} \|u\|_{E'(J)}.
\end{aligned}$$

The proof is complete. □

Note that in Lemma 4.10-4.11 we can replace $\mathcal{A}(J)$ by $\|T_{a,J}\|$.

Lemma 4.12. *Let $E \in \mathcal{M}$ be a strictly convex BFS and suppose that E' has AC-norm. Let u and v be constant over an interval $J = (c, d)$. Then $\mathcal{A}(J, u, v) \approx uv|J|$.*

Proof. From the Muckenhaupt condition we deduce that if $\widetilde{J} \subset J$ and $|\widetilde{J}|/|J| \geq 1/2$, then $\|\chi_{\widetilde{J}}\|_E \approx \|\chi_I\|_E$ and $\|\chi_{\widetilde{J}}\|_{E'} \approx \|\chi_I\|_{E'}$. Let $e \in (c, d)$, we have

$$\max \left\{ \sup_{t \in (c,e)} \|\chi_{(c,t)}\|_{E'} \|\chi_{(t,e)}\|_E,\ \sup_{t \in (e,d)} \|\chi_{(e,t)}\|_{E'} \|\chi_{(t,d)}\|_E \right\} \approx |J|.$$

Using Theorem 2.3 and Lemma 4.9 we obtain

$$\mathcal{A}(I, 1, 1) \approx |J|.$$

Consequently

$$\mathcal{A}(J, u, v) = \sup_{\|f\|_{E(J)}=1} \inf_{\alpha \in \mathbb{R}} \left\| v \left(\int_a^x f(t)u)dt - \alpha \right) \right\|_{E(J)}$$

$$= uv \sup_{\|f\|_{E(J)}=1} \inf_{c\in\mathbb{R}} \left\| \left(\int_a^x f(t)dt - c \right) \right\|_{E(J)}$$
$$= uv\mathcal{A}(J,1,1) \approx uv|J|.$$

□

5. Estimates of s-numbers for T

Throughout this section we view T as a map from a BFS E to itself.

Lemma 5.1. *Let E be a strictly convex BFS space that fulfills condition* (2.1)*, let E' have AC-norm, and suppose that $u \in E'(I)$ and $v \in E(I)$. Let $a = a_0 < a_1 < ... < a_N = b$ be a sequence such that $\mathcal{A}(a_{i-1}, a_i) \le \varepsilon$ for $i = 2, ..., N$ and $\|T_{a,(a,a_1)}\| \le \varepsilon$. Then*

$$a_N(T) \le C\varepsilon.$$

Proof. Set $I_i = (a_{i-1}, a_i)$ and $Pf = \sum_{i=2}^N P_i f$, where

$$P_i f(x) = v(x)\chi_{I_i} \int_a^{e_i} f(t)u(t)dt,$$

and e_i is a number obtained from Lemma 4.9 for which

$$\mathcal{A}(I_i) = \min_{x\in I_i} \|T_{x,I_i}|E(I_i) \to E(I_i)\| \approx \|T_{e_i,I_i}|E(I_i) \to E(I_i)\|.$$

Note that $\operatorname{rank}P \le N-1$; using Lemma 4.9 we obtain

$$\begin{aligned}
\|(T-P)f\|_E &= \|\chi_{I_1}T_{a,I_1}f + \sum_{i=2}^N (Tf - P_i f)\chi_{I_i}\|_E \\
&= \|\chi_{I_1}T_{a,I_1}f + \sum_{i=2}^N \chi_{I_i}T_{e_i,I_i}f\|_E \\
&\le C\|\{\|\chi_{I_1}T_{a,I_1}f\|_E, \|\chi_{I_i}T_{e_i,I_i}f\|_E\}\|_l \\
&\le C\max\{\|T_{a,I_1}\|, \mathcal{A}(I_2), ..., \mathcal{A}(I_N)\}\|\{\|f\chi_{I_i}\|_E\}\|_l \\
&\le C_1\varepsilon\|f\|_E.
\end{aligned}$$

□

Lemma 5.2. *Let E be a strictly convex BFS satisfying condition* (2.1) *. Let E^* be strictly convex and suppose that E' has AC-norm. Let $u \in E'(I)$ and $v \in E(I)$. Let $a = a_0 < a_1 < ... < a_N = b$ be a sequence such that $\mathcal{A}(a_{i-1}, a_i) \ge \varepsilon$ for $i = 2, ..., N$ and $\|T_{a,(a,a_1)}\| \ge \varepsilon$. Then*

$$i_N(T) \ge C\varepsilon.$$

Proof. The argument here is similar to the proof of Lemma 6.13 of [19] (which dealt with the case when E is a Lebesgue space), but we give full details for the convenience of the reader. Set $I_i = (a_{i-1}, a_i)$ $(i = 1, ..., N)$. From Lemma 4.9 it follows that there is $e_i \in I_i$ such that

$$\mathcal{A}(I_i) = \min_{x\in I_i} \|T_{x,I_i}|E(I_i) \to E(I_i)\| \approx \|T_{e_i,I_i}|E(I_i) \to E(I_i)\|.$$

Note also that (see Lemma 4.4)

$$\begin{aligned}
\|T_{e_i,(a_{i-1},e_i)}|E((a_{i-1},e_i)) \to E((a_{i-1},e_i))\| &= \|T_{e_i,(e_i,a_i)}|E((e_i,a_i)) \to E((e_i,a_i))\| \\
&\approx \|T_{e_i,I_i}|E(I_i) \to E(I_i)\|.
\end{aligned}$$

Since $T_{e_i,(a_{i-1},e_i)}$ and $T_{e_i,(e_i,a_i)}$ are compact operators there exist functions f_i^1, f_i^2 such that

$$\begin{aligned}&\operatorname{supp} f_i^1 \subset (a_{i-1}, e_i), \quad \operatorname{supp} f_i^2 \subset (e_i, a_i), \quad \|f_i^1\|_E = \|f_i^2\|_E = 1,\\ &\|T_{e_i,(a_{i-1},e_i)}|E((a_{i-1}, e_i)) \to E((a_{i-1}, e_i))\| = \|T_{e_i,(a_{i-1},e_i)} f_i^1\|_{E(e_i,(a_{i-1},e_i))}\end{aligned}$$

and

$$|T_{e_i,(e_i,a_i)}|E((e_i, a_i)) \to E((e_i, a_i))\| = |T_{e_i,(e_i,a_i)} f_i^2\|_{E((e_i,a_i))}.$$

Define $J_1 = (a_0, e_1) = (e_0, e_1)$, $J_i = (e_{i-1}, e_i)$ for $i = 2, ..., N$ and $J_{N+1} = (e_{N-1}, b)$. We introduce functions

$$\begin{aligned}&g_1(x) = f_1^1(x)\chi_{(e_0,e_1)}(x),\\ &g_i(x) = (c_i f_{i-1}^2(x)\chi_{(e_{i-1},a_{i-1})}(x) + d_i f_i^1(x)\chi_{(a_{i-1},e_i)}(x)) \quad \text{for} \quad i = 2, ..., N\end{aligned}$$

and

$$g_{N+1}(x) = f_N^2(x)\chi_{J_N}(x).$$

For these functions we have

$$\frac{\|T_{e_{i-1},J_i} g_i\|_{E((e_{i-1},a_{j-1}))}}{\|g_i\|_{E((e_{i-1},a_{j-1}))}} \geq C\varepsilon$$

and

$$\frac{\|T_{e_i,J_i} g_i\|_{E((a_{i-1},e_j))}}{\|g_i\|_{E((a_{i-1},e_j))}} \geq C\varepsilon \quad \text{for} \quad i = 1, ..., N+1.$$

We can see that $T_{e_{i-1},J_i} g_i$ and $T_{e_i,J_i} g_i$ do not change sign on (e_{i-1}, a_{i-1}) and (a_{i-1}, e_i) respectively. Since $T_{e_{i-1},J_i} g_i(x)$ and $T_{e_i,J_i} g_i(x)$ are continuous function we can choose constants c_i and d_i such that

$$T_{e_{i-1},J_i} g_i(a_{i-1}) = T_{e_i,J_i} g_i(a_{i-1}) > 0$$

and $\|g_j\|_{E(J_i)} = 1$. Then we can see that $\operatorname{supp}(Tg_i) \subset J_i$, $i = 2, ..., N$.

Note that

$$\begin{aligned}\frac{\|Tg_i\|_{E(J_i)}}{\|g\|_{E(J_i)}} &= \frac{\|T_{e_{i-1},(e_{i-1},a_{i-a})} g_i \chi_{(e_{i-1},a_{i-a})} + T_{e_i,(a_{i-1},e_i)} g_i \chi_{(a_{i-1},e_i)}\|_{E(J_i)}}{\|g\|_{E(J_i)}}\\ &\approx \frac{\|\{\|T_{e_{i-1},(e_{i-1},a_{i-a})} g_i\|_{E((e_{i-1},a_{i-a}))}, \|T_{e_i,(a_{i-1},e_i)} g_i\|_{E((a_{i-1},e_i)}\}\|_l}{\|g\|_{E(J_i)}}\\ &\geq C_1\varepsilon \quad \text{for} \quad i = 2, ..., N. \qquad (5.1)\end{aligned}$$

Since E and E^* are strictly convex BFS, given any $x \in E\backslash\{0\}$, there is a unique element of E^*, here written as $\widetilde{J}_E(x)$, such that $\|\widetilde{J}_X(x)\|_{X^*} = 1$ and $< x, \widetilde{J}_E(x) >$ $= \|x\|_E$. Note that for all $x \in E\backslash\{0\}$, $\widetilde{J}_E(x) = \operatorname{grad}\|x\|_E$, where $\operatorname{grad}\|x\|_E$ denotes the Gâteaux derivative of $\|\cdot\|_E$ at x (see [19]).

Denote by l the discrete Banach function space corresponding to the partition J_i, $i = 1, ..., N+1$ of the interval I. The maps $A : l \to E$ and $B : E \to l$ are defined by:

$$A(\{d'_i\}_{i=1}^N) = \sum_{i=1}^{N+1} d'_i g_i(x)$$

$$Bg(x) = \left\{ \frac{< g\chi_{J_i}, \widetilde{J}_E(Tg_i) >}{\|Tg_i\|_{E(J_i)}} \right\}_{i=1}^{N+1}.$$

Since $< Tg_i, \widetilde{J}_E(Tg_i) >= \|Tg_i\|_E$,

$$BTA(\{d_i\}_{i=1}^{N+1}) = \{d_i\}_1^{N+1}.$$

Observe that $\|B : E \to l\|$ is attained only for functions of the form

$$g(x) = \sum_{i=1}^{N+1} c_i' Tg_i(x),$$

Using (5.1) we obtain

$$\|g\|_E \geq C_2\varepsilon \|\{c_i'\}_{i=1}^{N+1}\|_l$$

and then

$$\sup_{\|f\|_E \leq 1} \|Bf\|_l = \sup_{\|g\|_E \leq 1} \|B(\sum_{i=1}^{N+1} c_n' Tg_i(x))\|_l = \sup_{\|g\|_E \leq 1} \|\{c_i'\}_{i=1}^{N+1}\|_l \leq C_2/\varepsilon.$$

From

$$\|A(\{d_i'\})_{i=1}^{N+1}\|_E \approx \|\{\|d_i' g_i\|_{E(J_i)}\}\|_l = \|\{d_i'\}\|_l$$

it follows that $\|A : l \to E\| \approx 1$. Thus

$$i_N(T) \geq \|A\|^{-1}\|B\|^{-1} \geq C_3\varepsilon.$$

□

Note that in the formulation of Lemmas 5.1 and 5.2 instead $\mathcal{A}$ we may use $\widehat{\mathcal{A}}$.

Let E be a BFS satisfying condition (2.1), let E' have AC-norm, and suppose that $u \in E'(I)$ and $v \in E(I)$. Note that for sufficiently small $\varepsilon > 0$ there are $c, d \in (a, b)$ for which $\widehat{\mathcal{A}}(c, b) = \varepsilon$ and $\|T_{a,(a,d)}\| = \varepsilon$. Indeed, since T is compact, there exists a positive integer $N(\varepsilon)$ and points $a = a_0 < a_1 < .... < a_{N(\varepsilon)} = b$ with $\widehat{\mathcal{A}}(a_{i-1}, a_i) = \varepsilon$ for $i = 2, ..., N(\varepsilon) - 1$, $\widehat{\mathcal{A}}(a_{N(\varepsilon)-1}, b) \leq \varepsilon$ and $\|T_{a,(a,a_1)}\| = \varepsilon$. The intervals $I_i = (a_{i)-1}, a_i)$, $i = 1, ..., N(\varepsilon)$ form a partition of I.

Lemma 5.3. *Let E be a BFS satisfying condition* (2.1)*, let E' have AC-norm, and suppose that $u \in E'(I)$ and $v \in E(I)$. Then the number $N(\varepsilon)$ is a non-increasing function of ε that takes on every sufficiently large integer value.*

Proof. As in the proof of Lemma 6.11 of [19], fix c, $a < c < b$. We have $\|T_{a,(a,c)}\| = \varepsilon_0 > 0$ and there is a positive integer $N(\varepsilon_0)$ and a partition $a = a_0 < a_1 < ... < a_{N(\varepsilon_0)} = b$ such that $\|T_{a,(a,a_1)}\| = \varepsilon_0$, $\widehat{\mathcal{A}}(a_{i-1}, a_i) = \varepsilon_0$ for $i = 2, ..., N(\varepsilon_0) - 1$, $\widehat{\mathcal{A}}(a_{N(\varepsilon_0)-1}, b) \leq \varepsilon_0$. Let $d \in (a, c)$. According to Lemma 4.6, $\widehat{\mathcal{A}}(a, d) = \varepsilon_0' < \varepsilon_0$ and the procedure outlined above applied with ε_0' gives $\infty > N(\varepsilon_0') \geq N(\varepsilon_0)$. By continuity of $\widehat{\mathcal{A}}(c, \cdot)$ and $\|T_{a,(a,\cdot)}\|$, there exists $d \in (a, c)$ such that $N(\varepsilon_0') > N(\varepsilon_0)$. If $N(\varepsilon_0') = N(\varepsilon_0) + 1$, stop. Otherwise, define

$$\varepsilon_1 = \sup\{\varepsilon : \ 0 < \varepsilon < \varepsilon_0 \text{ and } N(\varepsilon) \geq N(\varepsilon_0) + 1\}.$$

We claim $N(\varepsilon_1) = N(\varepsilon_0) + 1$. Indeed suppose $N(\varepsilon_1) \geq N(\varepsilon_0) + 2$ and the partition $a = a_0 < ... < a_{N(\varepsilon_1)} = b$ satisfies $\|T_{a,(a,a_1)}\| = \varepsilon_1$ and $\widehat{\mathcal{A}}(a_i, a_{i+1}) = \varepsilon_1$ $i = 1, 2...., N(\varepsilon_1) - 1$ and $\widehat{\mathcal{A}}(a_{N(\varepsilon_1)-1}, a_{N(\varepsilon_1)}) \leq \varepsilon_1$. Decrease $a_{N(\varepsilon_1)-1}$ slightly to $a'_{N(\varepsilon_1)-1}$ so that $\widehat{\mathcal{A}}(a'_{N(\varepsilon_1)-1}, b) < \varepsilon_1$ and $\widehat{\mathcal{A}}(a_{N(\varepsilon_1)-2}, a'_{N(\varepsilon_1)-1}) > \varepsilon_1$, continuing the process to get a partition of (a, b) having $N(\varepsilon_1)$ intervals such that

$\|T_{a,(a,a_1)}\| > \varepsilon_1$, $\widehat{\mathcal{A}}(a'_{i-1}, a'_i) > \varepsilon$, $i = 2, ..., N(\varepsilon_1) - 1$ and $\widehat{\mathcal{A}}(a_{N(\varepsilon_1)-1}, b) < \varepsilon_1$. Taking $\varepsilon_2 \leq \min\{\|T_{a,(a,a_1)}\|, \widehat{\mathcal{A}}(a'_{i-1}, a'_i);\ i = 2, ..., N(\varepsilon_1 - 1)\}$ we obtain $\varepsilon_2 > \varepsilon_1$ and $N(\varepsilon_2) \geq N(\varepsilon_0) + 2$, a contradiction. An inductive argument completes the proof. □

From Lemma 5.3, Lemma 4.6 and continuity of $\widehat{\mathcal{A}}(c, \cdot)$ and $\|T_{a,(c,\cdot)}\|$ the next lemma follows.

Lemma 5.4. *Let E be a BFS satisfying condition* (2.1) *, let E' have AC-norm, and suppose that $u \in E'(I)$ and $v \in E(I)$. Then for each $N > 1$ there exist ε_N and a sequence $a = a_0 < a_1 < .... < a_N = b$ such that $\widehat{\mathcal{A}}(a_{i-1}, a_i) = \varepsilon_N$ for $i = 2, ..., N$ and $\|T_{a,(a,a_1)}\| = \varepsilon_N$.*

Combining Lemmas 5.1-5.4 we obtain the following theorem.

Theorem 5.5. *Let E be a strictly convex BFS satisfying condition* (2.1)*, let E^* be strictly convex and E' have AC-norm, and suppose that Let $\|u\chi_I\|_{E'(I)}\|v\chi_I\|_{E(I)} < \infty$. Then for each $N > 1$ there exist ε_N and a sequence $a = a_0 < a_1 < .... < a_N = b$ such that $\mathcal{A}(a_{i-1}, a_i) = \varepsilon_N$ for $i = 2, ..., N$ and $\|T_{a,(a,a_1)}\| = \varepsilon_N$ and*

$$a_N(T) \approx i_N(T) \approx \varepsilon_N.$$

6. Asymptotic results

Theorem 6.1. *Let E be a strictly convex BFS satisfying condition* (2.1) *and suppose it has AC-norm. Let E^* be strictly convex, let E' has AC-norm, and suppose that $u \in E'(I)$ and $v \in E(I)$. Then there exist constants $C_1 = C_1(E), C_2 = C_2(E) > 0$ such that for the map $T : E \to E$*

$$C_1 \int_a^b u(x)v(x)dx \leq \limsup_{n\to\infty} N\varepsilon_N \leq \limsup_{n\to\infty} N\varepsilon_N \leq C_2 \int_a^b u(x)v(x)dx$$

Proof. As in the proof of Theorem 6.3 of [19] we observe that for each $\eta > 0$ there exist nonnegative step functions u_η, v_η on I such that

$$\|u - u_\eta\|_{E'(I)} < \eta, \;\; \|u - v_\eta\|_{E(I)} < \eta.$$

We may suppose that

$$u_\eta = \sum_{j=1}^m \xi_j \chi_{W(j)}, \;\; v_\eta = \sum_{j=1}^m \eta_j \chi_{W(j)}$$

where $W(j)$ are closed subintervals of I with disjoint interiors and $I = \cup_{j=1}^m W(j)$.

Let N be an integer greater than 1. By Lemma 5.4 there exist $\varepsilon_N > 0$ and a sequence a_k, $k = 0, 1, ..., N$, such that $a_0 = a$, $a_N = b$ and

$$\widehat{\mathcal{A}}(I_i) = \varepsilon = \varepsilon_N \text{ for } i = 2, ..., N \text{ and } \|T_{a,I_1}\| = \varepsilon \text{ where } I_k = (a_{k-1}, a_k).$$

We have

$$\begin{aligned}\left|\int_I u_\eta(t)v_\eta(t)dt - \int_I uv\right| &\leq \int_I u(t)|v(t) - v_\eta(t)|dt + \int_I |u(t) - u_\eta(t)|v_\eta(t)dt \\ &\leq \|u\|_{E'}\|v - v_\eta\|_E + \|u - u_\eta\|_{E'}\|v\|_E \\ &\leq \eta(\|u\|_{E'} + \|v\|_E + \eta). \qquad (6.1)\end{aligned}$$

Let $K = \{k > 1 :$ there exists j such that $I_k \subset W(j)\}$. Then $\#K \geq N - 1 - m$, and by Lemmas 4.10-4.12,

$$
\begin{aligned}
(N-1-m)\varepsilon &\le C_1 \sum_{k\in K} \widehat{\mathcal{A}}(I_k,u,v)\\
&\le C_2 \sum_{k\in K} \mathcal{A}(I_k,u,v)\\
&\le C_3 \sum_{k\in K} \Big\{\mathcal{A}(I_k,u_\eta,v_\eta)\\
&\qquad + (\mathcal{A}(I_k,u,v)-\mathcal{A}(I_k,u_\eta,v))\\
&\qquad + (\mathcal{A}(I_k,u_\eta,v)-\mathcal{A}(I_k,u_\eta,v_\eta))\Big\}\\
&\le C_4 \sum_{j} \Big\{|\xi_j||\eta_j||W(j)|\\
&\qquad + \|u-u_\eta\|_{E'(W(j))}\|v\|_{E(W(j))}\\
&\qquad\quad + \|v-v_\eta\|_{E(W(j))}\|u_\eta\|_{E'(W(j))}\Big\}\\
&\le C_4\Big(\int_I u_\eta(t)v_\eta(t)dt + \eta\|v\|_E + \eta(\|u\|_{E'}+\eta)\Big).
\end{aligned}
$$

By (6.1)we conclude that

$$
\limsup_{N\to\infty} N\varepsilon_N \le C_4\Big(\int_I u(t)v(t)dt + 2\eta\|v\|_E + 2\eta\|u\|_{E'} + \eta^2\Big)
$$

and then

$$
\limsup_{n\to\infty} N\varepsilon_N \le C_4 \int_I u(t)v(t)dt.
$$

To prove the opposite inequality we add the end-points of the intervals $W(j)$, $j=1,2,...,m$ to the a_k, $k=0,1,...,N$, to form the partition $a=e_0<...<e_n=b$, say, where $n\le N+1+m$. Note that each interval $J_i=(e_k,e_{k+1})$ is a subinterval of some $W(j)$ and hence u_η, v_η have constant values on each J_i. Thus

$$
\begin{aligned}
\int_I u_\eta v_\eta dt &= \int_{I_1} u_\eta v_\eta dt + \int_{I\setminus I_1} u_\eta v_\eta dt\\
&\le C_5\Big(\sum_{J_i\subset I_i} \|T_{a,J_i,u_\eta,v_\eta}\| + \sum_{J_i\not\subset I_i} \mathcal{A}(J_i,u_\eta,v_\eta)\Big).
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&\sum_{J_i\not\subset I_i} \mathcal{A}(J_i,u_\eta,v_\eta)\\
&\quad\le \sum_{J_i\not\subset I_i} \Big\{\mathcal{A}(J_i,u.v) + (\mathcal{A}(J_i,u_\eta.v)-\mathcal{A}(J_i,u.v))\\
&\qquad\quad + (\mathcal{A}(J_i,u_\eta.v_\eta)-\mathcal{A}(J_i,u_\eta.v))\Big\}\\
&\quad\le \sum_{J_i\not\subset I_i} \Big\{\mathcal{A}(J_i,u.v) + \|u-u_\eta\|_{E'}\|v\|_E + \|u_\eta\|_{E'}\|v_\eta-v\|_E\Big\};
\end{aligned}
$$

analogously for $\|T_{a,J,u_\eta}, v_\eta\|$ we have

$$\begin{aligned}\sum_{J_i\subset I_i} &\|T_{a,J_i,u_\eta,v_\eta}\| \\ &\le \sum_{J_i\subset I_i}\Big\{\|T_{a,J_i,u,v}\| + (\|T_{a,J_i,u_\eta,v}\| - \|T_{a,J_i,u,v}\|) \\ &\qquad + (\|T_{a,J_i,u_\eta,v_\eta}\| - \|T_{a,J_i,u_\eta,v}\|)\Big\} \\ &\le \sum_{J_i\subset I_i}\Big\{\|T_{a,J_i,u,v}\| + \|u-u_\eta\|_{E'}\|v\|_E + \|u_\eta\|_{E'}\|v_\eta - v\|_E\Big\}.\end{aligned}$$

Hence, from $\|T_{a,J,u,v}\| \le \varepsilon$ and $\mathcal{A}(J_i,u,v) \le C_5\varepsilon$

$$\int_I u(t)v(t)dt \le C_6((N+1+m)\varepsilon + 3\eta\|v\|_E + \eta(3\|u\|_{E'} + \eta))$$

and since $\eta > 0$ is arbitrary the theorem follows. □

Proof of Theorem 2.8 Combining Theorem 5.5 and Theorem 6.1 we obtain the proof of Theorem 2.8. □

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